Variational estimates for martingale transforms

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Rough paths

Definition

A *p-rough path*, $2 < p < 3$, is a pair

$$X : [0, \infty) \to H, \quad \mathbb{X} : \Delta = \{(s,t) \mid 0 \le s < t < \infty\} \to H \otimes H$$

such that $X \in V^p_{loc}$, $\mathbb{X} \in V^{p/2}_{loc}$, and for $s < t < u$

$$\mathbb{X}_{s,u} = \mathbb{X}_{s,t} + \mathbb{X}_{t,u} + (X_u - X_t) \otimes (X_t - X_s). \qquad \text{(Chen's relation)}$$

p-variation:

$$V^p X = \sup_{l_{max}, u_0 < \dots < u_{l_{max}}} \Big(\sum_{l=1}^{l_{max}} |X_{u_l} - X_{u_{l-1}}|^p \Big)^{1/p},$$

$$V^p \mathbb{X} = \sup_{l_{max}, u_0 < \dots < u_{l_{max}}} \Big(\sum_{l=1}^{l_{max}} |\mathbb{X}_{u_{l-1}, u_l}|^p \Big)^{1/p}.$$

How to check the conditions $X \in V^p$ and $\mathbb{X} \in V^{p/2}$?

Rough path lifts of martingales

Let $M = (M_t)$ be a (Hilbert space valued) càdlàg martingale. Let

$$\mathbb{M}_{s,t} := \int_{(s,t]} (M_{u-} - M_s) \otimes dM_u.$$

Then, a.s., the pair $(M, \mathbb{M})$ is a p-rough path for any $p > 2$.

- Chen's relation – from Itô integration
- Bound for $V^p M$: Lépingle 1976.
- Bounds for $V^{p/2}\mathbb{M}$:
  - M Brownian motion: Lyons 1998
  - M has continuous paths: Friz+Victoir 2006
  - M dyadic: Do+Muscalu+Thiele 2010,
  - M has càdlàg paths: Chevyrev+Friz 2017, Kovač+ZK 2018.

There exist rough path lifts of over processes, e.g. Lévy processes.
Q: what is the appropriate generality for these lifting results?
How to incorporate e.g. fractional Brownian motion?

Joint rough path lifts

All martingales and processes are adapted, càdlàg, Hilbert space valued.

Theorem (Friz+ZK 2020+)

Let $M = (M_t)$ be a càdlàg martingale and $(X, \mathbb{X})$ a deterministic càdlàg p-rough path ($2 < p < 3$). Then, a.s., the pair of processes

$$\begin{pmatrix} X \\ M \end{pmatrix}, \begin{pmatrix} \mathbb{X} & \int X_{u-} \otimes dM_u \\ \int M_{u-} \otimes dX_u & \int M_{u-} \otimes dM_u \end{pmatrix}$$

is a p-rough path.

New in this result:

- Variational estimates for Itô integrals $\int X \, dM$,
- existence of and estimates for integrals $\int M \, dX$.

The proof also recovers existence of Itô integrals and estimates for $\mathbb{M} = \int M \, dM$ from previous slide.

Martingale transforms

Let $(f_n)_{n\in\mathbb{N}}$ be a discrete time adapted process and $(g_n)_{n\in\mathbb{N}}$ a discrete time martingale. Define *paraproduct*

$$\Pi_{s,t}(f,g) := \sum_{s<j\le t} (f_{j-1} - f_s) dg_j, \quad dg_j = g_j - g_{j-1}.$$

Martingale in t variable, discrete version of area integral.

Theorem (Main estimate)

Let $1 \le p \le \infty$, $0 < q_1 \le \infty$, $1 \le q_0 < \infty$. Define q by $1/q = 1/q_0 + 1/q_1$ and suppose $1/r < 1/2 + 1/p$. Then, with $\|g\|_{L^q} = (\mathbb{E}|g|^q)^{1/q}$, $Sg = [g]^{1/2}$,

$$\left\| V^r \Pi(f,g) \right\|_{L^q} \lesssim \sup_\tau \left\| \Big(\sum_k \big(\sup_{\tau_{k-1}<j\le\tau_k} |f_{j-1} - f_{\tau_{k-1}}| \big)^p \Big)^{1/p} \right\|_{L^{q_1}} \|Sg\|_{L^{q_0}}.$$

The supremum is taken over increasing sequences of stopping times $\tau = (\tau_k)$.

- If f is a martingale, $p = 2$, $1 \le q_1 < \infty$, then by BDG inequality the RHS is $\lesssim \|Sf\|_{L^{q_0}} \|Sg\|_{L^{q_0}}$. In this case, any $r > 1$ works.
- For general f, RHS is $\le \|V^p f\|_{L^{q_0}} \|Sg\|_{L^{q_0}}$ and $r = p/2$ works.

Discrete approximation of adapted processes

Definition

An *adapted partition* $\pi = (\pi_j)_j$ is an increasing sequence of stopping times. Adapted partitions are ordered by a.s. inclusion of the sets $\{\pi_j \mid j \in \mathbb{N}\}$. The set of adapted partitions is directed, so lim_π makes sense. For an adapted partition π, let

$$\lfloor t, \pi \rfloor := max\{s \in \pi \mid s \leq t\}, \quad f_t^{(\pi)} := f_{\lfloor t,\pi \rfloor}.$$

Lemma

If $f \in L^q(V^p)$ for some $q > 0$ and $p > 1$, then

$$\lim_\pi f^{(\pi)} = f \quad in \quad L^q(V^{\tilde{p}})$$

for any $\tilde{p} \in (p, \infty) \cup \{\infty\}$.

Proof.

Given $\epsilon > 0$, consider the adapted partition

$$\pi_0 := 0, \quad \pi_{j+1}(\omega) := inf\Big\{t > \pi_j(\omega) \,\Big|\, |f_t - f_{\pi_j(\omega)}|(\omega) > \epsilon\Big\}.$$

□

Discrete approximation of Itô integrals

The Itô integral of the discretized process $f^{(\pi)}$ is given by

$$\int_0^T f^{(\pi)}_{u-}\, dM_u = \sum_{j:\pi_j \leq T} f_{\pi_{j-1}} (M_{\pi_j} - M_{\pi_{j-1}}), \quad T \in \pi.$$

The RHS is a martingale transform, to which our main estimate applies. Since it converges to the Itô integral, we get the same estimate for it:

$$\|V^r \Pi(f,g)\|_{L^q} \lesssim \|V^p f\|_{L^{q_1}} \|Sg\|_{L^{q_0}},$$

where $1/r > 1/2 + 1/p$ and

$$\Pi(f,g)_{s,t} = \int_{(s,t]} (f_{u-} - f_s)\, dg_u.$$

In fact, the discrete estimate gives more: the discrete approximations are a *Cauchy net* in the space $L^q(V^r)$, so we also reprove the existence of the Itô integral.

Stopping time reduction

f adapted process, g martingale
Martingale transform: $\Pi_{s,t}(f,g) = \sum_{s<j\le t}(f_{j-1} - f_s)dg_j$
Square function: $Sg = [g]^{1/2}$, Hölder exponents: $1/q = 1/q_0 + 1/q_1$.

Theorem (Main estimate)

Suppose $1/r < 1/p + 1/2$. *Then*

$$\|V^r\Pi\|_{L^q(\Omega)} \lesssim \|V^p f\|_{L^{q_1}(\Omega)} \|Sg\|_{L^{q_0}(\Omega)}.$$

The V^r norm is estimated as follows.

Lemma

Let $(\Pi_{s,t})_{s\le t}$ *be a càdlàg adapted sequence with* $\Pi_{t,t} = 0$ *for all* t.
Then, for every $0 < \rho < r < \infty$ *and* $q \in (0,\infty]$, *we have*

$$\|V^r\Pi\|_{L^q} \lesssim \sup_{\tau} \Big\| \Big(\sum_{j=1}^{\infty} \big(\sup_{\tau_{j-1}\le t<t'\le\tau_j} |\Pi_{t,t'}| \big)^{\rho} \Big)^{1/\rho} \Big\|_{L^q}, \tag{1}$$

where the supremum is taken over all adapted partitions τ.

Stopping time construction

For simplicity, we consider processes $\Pi_{s,t} = X_t - X_s$.
Let $V_n^\infty := sup_{n'' \le n' \le n} |X_{n''} - X_{n'}|$.
Construct stopping times with $m \in \mathbb{N}$:

$$\tau_0^{(m)} := 0, \quad \tau_{j+1}^{(m)} := inf\{t > \tau_j^{(m)} \mid |X_t - X_{\tau_j^{(m)}}| > 2^{-m} V_t^\infty / 10\}.$$

Then

$$(V^r X)^r \le C \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} |X_{\tau_j^{(m)}} - X_{\tau_{j-1}^{(m)}}|^r$$

$$\le C \sum_{m=0}^{\infty} (2^{-m} V_\infty^\infty)^{r-\rho} \sum_{j=1}^{\infty} |X_{\tau_j^{(m)}} - X_{\tau_{j-1}^{(m)}}|^\rho$$

Since $V^\infty \le V^r$, and assuming $V^r < \infty$, this implies

$$(V^r X)^\rho \le C \sum_{m=0}^{\infty} (2^{-m})^{r-\rho} \sum_{j=1}^{\infty} |X_{\tau_j^{(m)}} - X_{\tau_{j-1}^{(m)}}|^\rho.$$

Lépingle's inequality

Above stopping time argument first used in the following result.

Theorem (ZK 2019)

Let M be a martingale and w a positive random variable.
For $1 < p < \infty$ and $2 < r$, we have

$$\|V^r M\|_{L^p(w)} \leq C_{p,r} A_p(w)^{max(1,1/(p-1))} \|M\|_{L^p(w)},$$

where the A_p characteristic *is given by*

$$A_p(w) := \sup_{\tau \text{ stopping time}} \|\mathbb{E}(w \mid \mathcal{F}_\tau)\mathbb{E}(w^{-1/(p-1)} \mid \mathcal{F}_\tau)^{p-1}\|_{L^\infty(w)}$$

Classical Lépingle's inequality is the case $w \equiv 1$, $A_p(w) = 1$.
Weighted inequalities imply vector-valued inequalities.
For dealing with martingale transforms, we use vector-valued BDG inequalities that follow from weighted inequalities by Osękowski.

Sketch of proof of the main estimate

f adapted process, g martingale
Martingale transform: $\Pi_{s,t}(f,g) = \sum_{s<j\le t}(f_{j-1} - f_s)dg_j$
Square function: $Sg = [g]^{1/2}$, exponents: $1/q = 1/q_0 + 1/q_1$, $1/\rho = 1/p + 1/2$.

For an adapted partition τ, want to show

$$\Big\|\Big(\sum_l \sup_{\tau_{l-1}\le t\le t'\le\tau_l} |\Pi_{t,t'}|^\rho\Big)^{1/\rho}\Big\|_{L^q(\Omega)} \lesssim \|V^p f\|_{L^{q_1}(\Omega)}\|Sg\|_{L^{q_0}(\Omega)}.$$

Simple case: $q_1 = q_0 = p = 2$, $q = \rho = 1$.

$$\|\sum_{j=1}^{\infty} \sup_{[\tau_{j-1},\tau_j]} |\Pi|\|_1 = \sum_{j=1}^{\infty}\| \sup_{[\tau_{j-1},\tau_j]} |\Pi|\|_1 \overset{BDG}{\lesssim} \sum_{j=1}^{\infty}\|S\Pi_{\tau_{j-1},\tau_j}\|_1$$

$$= \mathbb{E}\sum_{j=1}^{\infty}\Big(\sum_k |f_{k-1}^{(j)}|^2 |g_k^{(j)} - g_{k-1}^{(j)}|^2\Big)^{1/2} \quad (\text{here } f_t^{(j)} = f_{t\wedge\tau_j} - f_{t\wedge\tau_{j-1}})$$

$$\le \mathbb{E}\sum_{j=1}^{\infty}(f_*^{(j)})\Big(\sum_k |g_k^{(j)} - g_{k-1}^{(j)}|^2\Big)^{1/2} \le \Big(\mathbb{E}\sum_{j=1}^{\infty}(f_*^{(j)})^2\Big)^{1/2}\Big(\mathbb{E}\sum_{j=1}^{\infty}\sum_k |g_k^{(j)} - g_{k-1}^{(j)}|^2\Big)^{1/2}$$

If one of the conditions $q_1 = p$, $q = \rho$, $q_0 = 2$ fails, things get more tricky.

Integration by parts

$(X, \mathbb{X})$ rough path, M martingale

So far we have estimated $\int X\,dM$ and $\int M\,dM$.

Next, we want to construct and estimate $\Pi(M,X) = \int M\,dX$.

We do this by partial integration:

$$\Pi(M,X) := \delta M \delta X - \Pi(X,M) - \delta[X,M].$$

The bracket is given by

$$[X,M]_T = \sum_{u \leq T} \Delta X_u \Delta M_u, \quad \Delta M_u = M_u - M_{u-}.$$

Variation norm estimate for the bracket:

$$\|V^r[X,M]\|_{L^q} \overset{\text{stopping}}{\lesssim} \Big\| \Big(\sum_{j=1}^{\infty} \big(\sup_{\tau_{j-1} < t < t' \leq \tau_j} |\delta[X,M]_{t,t'}| \big)^{\rho} \Big)^{1/\rho} \Big\|_{L^q}$$

$$\overset{\text{vector BDG}}{\lesssim} \Big\| \Big(\sum_{j=1}^{\infty} \big(\sum_{\tau_{j-1} < u \leq \tau_j} |\Delta X_u \Delta_u M|^2 \big)^{\rho/2} \Big)^{1/\rho} \Big\|_{L^q}$$

$$\overset{\text{Hölder}}{\leq} V^p X \cdot \Big\| \Big(\sum_{j=1}^{\infty} \sum_{\tau_{j-1} < u \leq \tau_j} |\Delta_u M|^2 \Big)^{1/2} \Big\|_{L^q}$$