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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 2

The first two exercise sheets will only use material you should be familiar with already. Some of it is covered and recalled by the first three lectures. These two sheets are not compulsory but the points can be counted towards your final score of the necessary 50% to get admitted to the exams.

Exercise 5. (Factor rings of polynomial rings, 2 points)

(i) Given a field k, and an element $a \in k$. Show that there is a ring isomorphism $k[x]/(x-a) \cong k$ where $(x-a) \subset k[x]$ is the principal ideal generated by $x-a$.

(ii) Let k be a field and $f \in k[x]$ a polynomial of degree d. Show that $k[x]/(f)$ is a k-vector space of dimension d.

(iii) Let k be a field and $a := (a_1, \ldots, a_n) \in k^n$. Show that the evaluation map

$$\varphi_a : k[x_1, \ldots, x_n] \to k, \ f(x_1, \ldots, x_n) \mapsto f(a_1, \ldots, a_n)$$

is a surjective ring homomorphism. Give generators for the kernel and show that it is a maximal ideal.[1]

Exercise 6. (Quotient modules, 3 points)

Use the corresponding facts from group theory to prove the following assertions:

(i) Let $M_1 \subset M_2 \subset M$ be A-submodules. Then there exists a natural isomorphism of A-modules:

$$(M/M_1)/(M_2/M_1) \cong M/M_2.$$

(ii) Let $M_1, M_2 \subset M$ be A-submodules. Then there exists a natural isomorphism of A-modules:

$$(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2).$$

Exercise 7. (Module homomorphisms, 4 points)

(i) Let M and N be A-modules. Show that the group $\mathrm{Hom}_A(M, N)$ of all A-module homomorphisms $f\colon M \to N$ has a natural structure of A-module.

(ii) Imitate the corresponding argument for vector spaces and show that there exists a natural isomorphism of A-modules:

$$M \cong \mathrm{Hom}_A(A, M).$$

(iii) Think of an example of an A-module $M \neq 0$ with $\mathrm{Hom}_A(M, A) = 0$.

(iv) Rephrase the statements for $A = \mathbb{Z}$ in terms of abelian groups.

Exercise 8. (Spectrum of a ring, 3 points)

Describe $\mathrm{Spec}(A)$ and $\mathrm{MaxSpec}(A)$ of the following rings $A = \mathbb{F}_p[x]$, $A = k[x]/(x^3)$, and $A = k[[x]]$, where k is an arbitrary field.

Solutions to be handed in before Monday April 20, 4pm.

[1] Notions like prime and maximal ideals will be recalled in the lecture on Thursday.