'Cones of products'

Summer term 2014, Tuesday 2-4pm, 0.011

We should try to learn everything that is known (and not known) about the nef cone on (symmetric) products of curves and surfaces. It would give us the opportunity to learn or revise some basic techniques, e.g. Seshadri constants, and get acquainted with some of the main conjectures. Seemingly basic questions are wide open. E.g. Kollár asked whether for a general curve C of genus ≥ 2 the diagonal Δ is the only irreducible curve in $C \times C$ with negative self-intersection (which corresponds to the effective cone being open on one side). The proposal emphasizes the nef cone, but other cones, e.g. the movable cone, are equally interesting.

The seminar could split naturally in two parts: curves and surfaces. These two parts should obviously be related but it seems that the actual implications between the two have not really been addressed in the literature.

The nef cone of $C \times C$ or of $S^2(C)$ is not completely understood, but there are various conjecture and questions. The shape of the cone depends on the curve and no uniform behavior should be expected. However, for higher products C^n the situation becomes more tractable.

For surfaces, the question becomes a question about the nef cone of the Hilbert scheme $\mathrm{Hilb}^n(S)$ of a surface S. Here, one could go towards conjectures and results of Hassett and Tschinkel with the recent progress by Bayer and Macri and others.

For further information or if you want to give a talk in the seminar, please contact one of us huybrech@, lazic@, or schnell@. The **first talk will be on April 15.** The seminar takes place every Tuesday, 2-4pm in 0.011.

The following is open for discussion. We may decide to shift the emphasize or dwell longer on certain aspects. In particular, the last four slots are left open for the time being. Presumably, some of the earlier talks will take more time and possibly we wish to include other material.

15 April: Introduction and survey.
(Speaker: Daniel Huybrechts)

22 April: Kouvidakis's theorem. Follow [14, Ch. 1.5] to prove that for a simple branched covering $C \to \mathbb{P}^1$ of degree $d \leq [\sqrt{g}] + 1$ one has $t(C) = \frac{g}{d-1}$, which describes the nef cone of $S^2(C)$ (see [14, Ex. 1.5.14]). Here, $t(C)$ is the minimum of all t with $(f_1 + f_2) - t\Delta$ being nef on $S^2(C)$. As a corollary one can prove that for the general curve $\sqrt{g} \leq t(C) \leq \frac{g}{[\sqrt{g}]}$. It is conjectured that $\sqrt{g} = t(C)$ for the general curve, but note that $d \leq [\sqrt{g}] + 1$ makes C special, see comments in [19] where it is also shown that in this range $C \times C$ is unstable.
(Speaker: Stefan Schreieder)

29 April: Vojta's method yields nef classes of the form $a_1 f_1 + a_2 f_2 + \Delta$ on $C \times C$ (needed for his proof of the Mordell conjecture). These classes are not symmetric, so do not come from $S^2(C)$. The construction has recently been slightly improved in [18]. (Speaker: Emanuel Reinecke)

13 May: Higher symmetric products. As it turns out, the nef cone of $S^k(C)$ of a general (e.g. contained in a K3 surface) curve of genus $g = 2k$ can be computed. This was done in [17]. The proof relies on Voisin's results on Green's conjecture, but we should treat this essentially as a black box (but we have learned most of it in Lazarsfeld's Felix Klein lectures).
(Speaker: Ch. Schnell)

20 May: How to use the Nagata conjecture. The Nagata Conjecture predicts the Seshadri constant at g general points in $\mathbb{P}^2$ and is essentially open. As shown in [9] and later again in [20] the Nagata conjecture can be used to describe the nef cone of $S^2(C)$ for a very general curve C of genus $g \geq 9$. In this talk one should first recall the notion of the Seshadri constant and state the Nagata conjecture. The standard reference is of course [14]. One could mention the classical result by Ein and Lazarsfeld about the behavior of the Seshadri constant in a very general point of the surface. Finally, an outline of the proof in [20] should be presented.
(Speaker: Luigi Lombardi)

27 May: First positivity results on $\mathrm{Hilb}^n(S)$ for projective surfaces S. Göttsche's criterion for nefness gives a bound for the natural line bundes $S^n(L) \otimes \mathcal{O}(-\delta)$. See [6] and [7].
(Speaker: Andreas Krug)

3 June: Hassett–Tschinkel conjecture. In [10, 11] one finds a conjecture about the effective cone of deformations of $\mathrm{Hilb}^n(S)$ for a K3 surface S. There are plenty of explicit and geometrically interesting examples, mostly for $n = 2$ and $\rho(S) = 1$. In [3] it was recently shown that the conjecture has to be modified for $n \geq 5$. The final result [3, Prop. 10.3] gives a complete description of the nef cone of $\mathrm{Hilb}^n(S)$ for $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ with $n \geq (H^2 + 6)/4$. (Note that the result says little about $\mathrm{Hilb}^2(S)$ or $S \times S$.) Here one could try to sketch the proof without going into the details of the proof of [3, Thm. 1.1] and stability conditions.)
(Speaker: Emanuele Macri)

3 June, 4:15pm: Kobayashi non-hyperbolicity of hyperkähler manifolds (Speaker: Ljudmila Kamenova (Stony Brook))

17 June: K3$^{[n]}$-type. The nef cone of deformations of $\mathrm{Hilb}^n(S)$ with S a K3 surface has been completely determined in [5] and [15]. We should take the results in [4] for granted and present one of the two proofs (or both).
(Speaker: Ulrike Rieß)

8 July: Hassett-Tschinkel conjecture; Kawamata-Morrison cone conjecture (after Markman-Yoshioka) (Speaker: Ulrike Riess; Daniel Huybrechts)

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