Boundary Control of Flows in Networks

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joint work with

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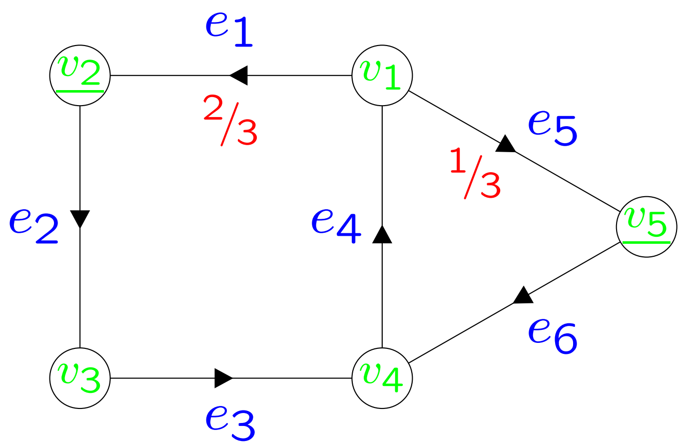
Minisymposium, DMV-Jahrestagung
18. September 2006

Outline of the Talk

- Flows in Networks
- Some Notions from Graph Theory
- The Mathematical Model
- Abstract Formulation
- Boundary Control of Linear Systems
- Boundary Control of Flows in Networks
- Outlook

Flows in Networks

Example 1



Here we have

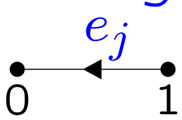
$$(\Phi_w^-)^\top = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{R}^{\mathsf{BC}}_{\mathsf{max}} = \left\{ (g_1, \ldots, g_6)^\top \in \mathsf{L}^1([0,1], \mathbb{C}^6) : 2\, g_5 = g_1 \right\}$$

$$= \mathcal{R}^{\mathsf{BC}}_i \iff i = 2 \text{ or } 5$$

Some Notions from Graph Theory

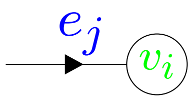
Consider

- $V = \{v_1, \ldots v_n\}$ = *vertices*
- $E = \{e_1, \ldots e_m\}$ = *edges*
- $e_j = [0, 1]\ \forall\ j$: 

Definition 2 (Graph Matrices)

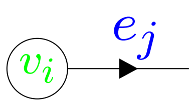
(i) $\Phi^+ = (\varphi^+_{ij})_{n\times m}$ = incoming incidence matrix

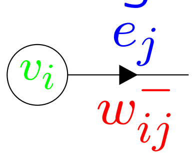
$$\varphi^+_{ij} = \begin{cases} 1 & \textit{if} \\ 0 & \textit{else} \end{cases}$$



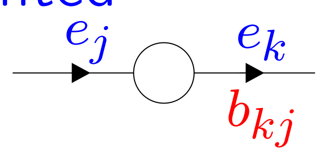
(ii) $\Phi^- = (\varphi^-_{ij})_{n\times m}$ = outgoing incidence matrix

$$\varphi^-_{ij} = \begin{cases} 1 & \textit{if} \\ 0 & \textit{else} \end{cases}$$



(iii) $\Phi^-_w = (w^-_{ij})_{n\times m}$ = weighted outgoing incidence matrix: 

(iv) $\mathbb{A} = (a_{ik})_{n\times n} = \Phi^+(\Phi^-_w)^\top$ = weighted adjacency matrix: 

(v) $\mathbb{B} = (b_{kj})_{m\times m} = (\Phi^-_w)^\top \Phi^+$ = weighted adjacency matrix of line graph: 

Remark 3 $\Phi^-(\Phi^-_w)^\top = I_{\mathbb{C}^n}$

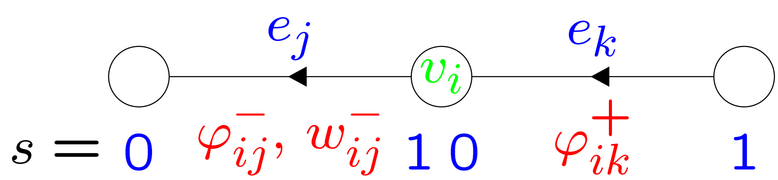
The Mathematical Model (simplest case)

- Transport Equation

(TE) $$\frac{\partial}{\partial t} u_j(t,s) = \frac{\partial}{\partial s} u_j(t,s)$$

- Boundary Condition (Kirchhoff law: out = in)

(BC) $$\varphi_{ij}^- u_j(t,1) = w_{ij}^- \sum_{k=1}^{m} \varphi_{ik}^+ u_k(t,0)$$



$(k = 1\ldots, m)$

- Initial Condition

(IC) $$u_j(0,s) = f_j(s)$$

where

$t \geq 0$ time variable

$s \in [0,1]$ space variable

$j = 1,\ldots,m$ (number of edges)

$i = 1\ldots n$ (number of vertices)

Abstract formulation

(TE), (BC), (IC) $\iff$ Abstract Cauchy Problem

(ACP) $$\begin{cases} \frac{d}{dt} u(t) = A\, u(t), \; t \geq 0 \\ u(0) = f \end{cases}$$

on X, where

- $X = \mathrm{L}^1([0,1], \mathbb{C}^m)$ = state space
- $A = \mathrm{diag}(\frac{d}{ds})_{m \times m}$ with domain (use Rem.3)

$$D(A) = \left\{ g \in \mathrm{W}^{1,1}([0,1], \mathbb{C}^m) : \begin{matrix} g(1) \in \mathrm{Rg}(\Phi_w^-)^\top \& \\ \Phi^- g(1) = \Phi^+ g(0) \end{matrix} \right\}$$
$$= \left\{ g \in \mathrm{W}^{1,1}([0,1], \mathbb{C}^m) : g(1) = \mathbb{B}\, g(0) \right\}$$

- $u(t) = u(t, \cdot)$, $f = (f_1, \ldots, f_m)^\top$

Remark 4 $A \sim$ Difference Equation

(ADE) $$v(t) = \begin{cases} \mathbb{B}\, v(t-1), & t > 1 \\ f(t), & t \in [0,1] \end{cases}$$

Corollary 5 *A generates the str.cont.semigroup*

$$[T(t)g](s) = \mathbb{B}^n g(t+s-n) \quad \textit{if } t+s \in [n, n+1)$$

where $\mathbb{B}^0 := 0$. In particular, $[T(n)\, g](s) = \mathbb{B}^n g(s)$

Boundary Control of Linear Systems

Now consider the Boundary Control System

$$(\mathrm{ACP_{BC}}) \qquad \begin{cases} \frac{d}{dt}u(t) = A_m u(t), & t \geq 0 \\ \boxed{Q\,u(t) = C\,w(t)} & t \geq 0 \\ u(0) = f \end{cases}$$

on X for

- Banach spaces X = *state*, ∂X = *boundary* and U = *control space*
- a closed, d.d. operator $A_m : D(A_m) \subseteq X \to X$
- a *boundary operator* $Q \in \mathcal{L}(D(A_m), \partial X)$
- a *control operator* $C \in \mathcal{L}(U, \partial X)$
- a control $w \in \mathrm{L}^1(\mathbb{R}_+, U)$ & I.C. $f \in X$

Remark 6 *(ACP)* $\iff$ *(ACP$_{BC}$)* *for*

- $A_m = \mathrm{diag}(\frac{d}{ds})_{m\times m}$ *with domain* $D(A_m) = \left\{g \in \mathrm{W}^{1,1}([0,1], \mathbb{C}^m) : g(1) \in \mathrm{Rg}(\Phi_w^-)^\top\right\}$
- $\partial X = \mathbb{C}^n$, $Qv = \Phi^- v(1) - \Phi^+ v(0)$, $C = 0$

Hypothesis 7

- $A := A_m|_{\ker Q}$ *generates* $(T(t))_{t\geq 0}$ *on* X
- $Q : D(A_m) \to \partial X$ *is surjective*

Lemma 8 (Greiner,'87)

(i) $Q_\lambda := (Q|_{\ker(\lambda - A_m)})^{-1} \in \mathcal{L}(\partial X, X)$

(ii) $D(A_m) = D(A) \oplus \ker(\lambda - A_m)$ *with projection* $P_\lambda := Q_\lambda Q \in \mathcal{L}(D(A_m))$ *onto* $\ker(\lambda - A_m)$

Proposition 9 (Variation of Const. Formula)

- *If $u(\cdot)$ solves (ACP$_{BC}$) then*

$$u(t) = T(t)f + \boxed{(\lambda - A_{-1}) \int_0^t T(t-s) Q_\lambda C\, w(s)\, ds}$$
$$=: T(t)f + \boxed{(\lambda - A_{-1})\, \mathcal{C}_t^\lambda\, w}$$
$$=: T(t)f + \mathcal{C}_t^{\mathsf{BC}}\, w$$

- *If $f \in D(A_m)$, $w \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, U)$ are suff.regular & $Qf = C\, w(0)$ (= compat.cond.) then*

$$u(t) := T(t)f + \mathcal{C}_t^{\mathsf{BC}}\, w$$

is the classical solution of (ACP$_{BC}$)

Proof. $u(t) = (I - P_\lambda)u(t) + P_\lambda u(t) \in D(A) + \ker(\lambda - A_m) \Rightarrow$
$\frac{d}{dt}u(t) = A_{-1}u(t) + (\lambda - A_{-1})Q_\lambda C\, w(t)$ □

Remarks 10 *(i)* $\mathcal{C}_t^\lambda$ = controllability map *of*

$$(\mathsf{cACP}_\lambda) \quad \begin{cases} \frac{d}{dt}v(t) = A\, v(t) + Q_\lambda C\, w(t), & t \geq 0 \\ v(0) = f \end{cases}$$

where $Q_\lambda C \in \mathcal{L}(U, X)$, *i.e.,* $v(t) = T(t)f + \mathcal{C}_t^\lambda\, w$

(ii) $\mathcal{C}_t^\lambda\, w \in D(A)$ *if*

- $D(A_m) \in \mathrm{Fav}(A)$ *or*
- $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, U)$

(iii) $\mathcal{C}_t^{\mathsf{BC}}$ = bound.control.map *is independ. of* λ

Theorem 11 *Let*

$$\mathcal{R}^{\mathsf{BC}} := \overline{\bigcup_{t \geq 0} \mathsf{Rg}(\mathcal{C}_t^{\mathsf{BC}})} = \text{app.reach.space of } (\mathsf{ACP}_{\mathsf{BC}})$$

$$\mathcal{R}^{\lambda} := \overline{\bigcup_{t \geq 0} \mathsf{Rg}(\mathcal{C}_t^{\lambda})} = \text{app.reach.space of } (\mathsf{cACP}_{\lambda})$$

Then

$$\mathcal{R}^{\mathsf{BC}} = \mathcal{R}^{\lambda} \quad \forall\, \lambda > \omega_0(A)$$

$= \textit{smallest closed } T(t)\textit{-inv. subspace of } X$ $\textit{containing } \mathsf{Rg}(Q_\lambda C) \textit{ for some/all } \lambda > \omega_0(A)$

$= \textit{smallest closed } R(\mu, A)\textit{-inv. subspace of } X$ $\textit{containing } \mathsf{Rg}(Q_\lambda C) \textit{ for some/all } \lambda > \omega_0(A)$

$$= \overline{\mathsf{lin}}\Big(\bigcup_{\lambda > w} \mathsf{Rg}(Q_\lambda C)\Big) \quad \forall\, w > \omega_0(A)$$

Remark 12 $\mathsf{Rg}(Q_\lambda) = \ker(\lambda - A_m)$ *hence*

$$\mathcal{R}^{\mathsf{BC}} \subseteq \overline{\mathsf{lin}}\Big(\bigcup_{\lambda > \omega_0(A)} \ker(\lambda - A_m)\Big) =: \mathcal{R}^{\mathsf{BC}}_{\mathsf{max}}$$

Problem 13 *When holds* $\mathcal{R}^{\mathsf{BC}} = \mathcal{R}^{\mathsf{BC}}_{\mathsf{max}}$ *?*

Boundary Control of Flows in Networks

Problem 14 *Let* $\mathcal{R}_i^{\mathsf{BC}} :=$ *set of states which can be approximately reached in any time by a control* w *acting only on* v_i $(1 \le i \le n)$

When holds $\mathcal{R}_i^{\mathsf{BC}} = \mathcal{R}_{\mathsf{max}}^{\mathsf{BC}}$ *?*

Here we have

- $U = \mathbb{C}$, $\quad C = C_i : U \to \mathrm{lin}\{v_i\} \subset \partial X = \mathbb{C}^n$

Moreover

- $\mathcal{R}_{\mathsf{max}}^{\mathsf{BC}} = \left\{ (\Phi_w^-)^\top g : g \in X \right\} \neq X = \mathrm{L}^1([0,1], \mathbb{C}^m)$

Theorem 15

$$\mathcal{R}_i^{\mathsf{BC}} = \mathcal{R}_{\mathsf{max}}^{\mathsf{BC}} \iff \mathrm{lin}\left\{ v_i, \mathbb{A} v_i, \ldots, \mathbb{A}^{n-1} v_i \right\} = \mathbb{C}^n$$

$$\iff$$

$$(\mathrm{cACP}_i) \qquad \begin{cases} \frac{d}{dt} x(t) = \mathbb{A}\, x(t) + v_i \cdot w(t), \ t \ge 0 \\ x(0) = f \end{cases}$$

is exactly controllable on $\mathbb{C}^n$

Proof. Use Cor.5 & Thm.11 $\square$

Outlook

- Different and space dependent velocities
- Absorption, scattering and sources
- Infinite networks
- Diffusion instead of flow
- ...