The transport Oka-Grauert principle for simple surfaces

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Inverse Problems in Analysis and Geometry

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Overview 1/2: Range characterisations in inverse problems

Inverse problems are typically posed in terms of a *forward operator*

$$\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}.$$

Often $\mathcal{F}^{-1}$ is not available, so we ask for injectivity, stability, ...

... & the range:

Problem: Characterise/understand the range $\mathcal{F}(\mathcal{X}) \subset \mathcal{Y}$.

Examples:

1. Helgason-Ludwig (1964): $\mathcal{F}$ = linear X-ray transform on $\mathbb{R}^n$ // range is characterised by moment conditions;
2. Pestov-Uhlmann (2004): $\mathcal{F}$ = linear X-ray transform on simple surface // range is parametrised by boundary operator;
3. Sharafutdinov (2011): $\mathcal{F}$ arising from Calderón problem on disk // elements of the range are related by conjugation;
4. Burago-Ivanov (2014): $\mathcal{F}$ = boundary distance map for Finsler metrics on n-ball // range is open under suitable perturbations;
5. **This talk:** $\mathcal{F}$ = non-Abelian X-ray transform on simple surface // nonlinear version of Pestov-Uhlmann result.

Overview 2/2: Connections to complex geometry

Common theme for some of these characterisations in 2D: Based on hard **transitivity theorem** with **complex geometric interpretation**.

	Transitivity theorem	Complex geometry
Calderón problem on the disk	any g is conformally flat	Riemann mapping theorem
Linear X-ray on simple surface	∃ "scalar holomorphic integrating factors"	$H^{0,1}_{\bar\partial}(Z) = 0$
Non-Abelian X-ray on simple surface	∃ "matrix holomorphic integrating factors"	Transport Oka-Grauert principle: $\mathfrak{M}(Z) = 0$
	$\Longleftrightarrow$ transitivity of a certain group action	We introduce a novel transport twistor space Z

Structure of talk: ●●●● → ●●●● (● $\rightsquigarrow$ Gabriel's talk)

Let (M, g) be a compact Riemannian surface with boundary ∂M. Assume that ∂M is strictly convex and that M is *non-trapping* ($\Rightarrow M \approx$ disk).

On $SM = \{(x, v) \in TM : g(v, v) = 1\}$ consider the **transport equation**

$$(X + \mathbb{A})R = 0 \text{ on } SM, \tag{TE}$$

with $X =$ *geodesic vector field* and $\mathbb{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ an *attenuation*.

Note: $R \in C^\infty(SM, \mathbb{C}^{n\times n})$ solves (TE), iff $\forall$ geodesics $\gamma \colon [0, \tau] \to M$,

$$\frac{d}{dt} R(\gamma(t), \dot\gamma(t)) + \mathbb{A} R(\gamma(t), \dot\gamma(t)) = 0. \tag{TE'}$$

Let $\partial_\pm SM = \{(x, v) \in SM : x \in \partial M, \pm g(v, \nu(x)) \geq 0\}$ = influx /outflux.

Definition

Let $R =$ unique solution of (TE) with $R|_{\partial_- SM} = \mathrm{Id}$, define:

$$C_{\mathbb{A}} = R|_{\partial_+ SM} \in C^\infty(\partial_+ SM, GL(n, \mathbb{C})) \quad \rightsquigarrow \quad \text{scattering data of } \mathbb{A};$$

$$\mathbb{A} \mapsto C_{\mathbb{A}} \quad \rightsquigarrow \quad \text{non-Abelian X-ray trafo.}$$

Examples:

- Scalar case ($n = 1$): $C_{\mathbb{A}} = \exp(I\mathbb{A})$, where $I =$ linear X-ray transform;
- Connections: If $\mathbb{A}(x, v) = A_x(v)$ for 1-form $A \in \Omega^1(M)$, then

$$C_A = \text{parallel transport of connection } d + A \text{ on } M \times \mathbb{C}^n;$$

- Polarimetric Neutron Tomography: If $\mathbb{A}(x, v) = \Phi(x) \in \mathfrak{so}(3)$, then

$$C_\Phi = \text{ spin rotation in } SO(3) \text{ of neutrons after traversing } \vec{B} \text{ field.}$$

Theorem (Paternain-Salo-Uhlmann 2012 & 2020)

Let (M, g) be simple (i.e. ∂M strictly convex, non-trapping & no conjugate points). Suppose $\mathbb{A}(x, v) = A_x(v) + \Phi(x)$ and $\mathbb{B} = B_x(v) + \Psi(x)$ are s.th.

$$C_{\mathbb{A}} = C_{\mathbb{B}}.$$

Then there exists a gauge $\varphi \in C^\infty(M, GL(n, \mathbb{C}))$ with $\varphi = \mathrm{Id}$ on ∂M and

$$\Phi = \varphi^{-1}\Psi\varphi, \quad A = \varphi^{-1}d\varphi + \varphi^{-1}B\varphi.$$

Theorem (B.-Paternain)

Let (M,g) be a simple surface and $q \in C^\infty(\partial_+ SM, U(n))$, then TFAE:

1. $q = C_{\mathbb{A}}$ *for some* $\mathfrak{u}(n)$*-valued* $\mathbb{A} = \Phi + A$*;*
2. q *lies in the range of a* **boundary operator**

$$P\colon C^\infty(\partial_+ SM, \mathbb{C}^{n\times n}) \supset D(P) \to C^\infty(\partial_+ SM, U(n)).$$

- Nonlinear version of Pestov-Uhlmann (2004);
- P defined in terms of Birkhoff factorisation; *morally* its domain is

$$D(P) \approx \begin{array}{c}\text{Hermitian metrics}\\ \text{on } \partial_+ SM \times \mathbb{C}^n\end{array} \approx \begin{array}{l}\text{Radiative/dispersive}\\ \text{degrees of freedom (DOF);}\end{array}$$

- Analogy with Ward correspondence by Mason (2006):

$$\begin{array}{c}\text{Solutions to}\\ \text{ASDYM}\end{array} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c}\text{Solitonic}\\ \text{DOF}\end{array} \right\} \times \left\{ \begin{array}{c}\text{Radiative/dispersive}\\ \text{DOF}\end{array} \right\}$$

- TOG principle: $\nexists$ nontrivial holomorphic vector bundles on Z.

$$\begin{aligned} \mathrm{Her}_+^n &= \text{Hermitian positive definite } n \times n \text{ matrices;} \\ \mathbb{G} &= \{F \in C^\infty(SM, GL(n,\mathbb{C})) : F, F^{-1} \text{ are fibrewise holomorphic}\}; \\ \alpha &= \text{scattering relation of } (M,g). \end{aligned}$$

Theorem (Symmetric Birkhoff factorisation)

For any $H \in C^\infty(SM, \mathrm{Her}_+^n)$ there exists $F \in \mathbb{G}$ such that $H = F^* F$.

How to generate elements in the range of $C^\infty(M, \mathfrak{u}(n)) \ni \Phi \mapsto C_\Phi$:

1. Start with $w \in D(P) := C_\alpha^\infty(SM, \mathrm{Her}_+^n)$;
2. extend to first integral $w^\sharp \in C^\infty(SM, \mathrm{Her}_+^n)$;
3. factor as $w^\sharp = F^* F$ (unique after requiring $F_0 = \mathrm{Id}$);
4. let $\Phi = -(XF)F^{-1} \in C^\infty(M, \mathfrak{u}(n))$, then

$$C_\Phi = Pw := F|_{\partial SM} \circ (F^{-1}|_{\partial SM} \circ \alpha) \quad \text{on } \partial_+ SM.$$

- To get the whole range, need to solve $(X + \Phi)F$ with $F \in \mathbb{G}$ ($\rightsquigarrow$ **HIF**);
- we prove existence of **HIF** using injectivity of I_Φ and Nash-Moser IFT.

Transport twistor space

We set up a correspondence for **any** orientable Riemannian surface:

$$(M,g) \quad \rightsquigarrow \quad \text{(degenerated) complex surface } Z;$$
$$\mathbb{A} \quad \rightsquigarrow \quad \text{holomorphic vector bundle over } Z.$$

Idea: Fill in the disks in SM and extend X to Cauchy-Riemann operator.

Theorem (The transport twistor space)

The 4-manifold $Z = \{(x,v) \in TM : g(v,v) \leq 1\}$ supports a **unique** *complex rank 2 distribution $D \subset T_{\mathbb{C}}Z$ with the following properties:*

1. *D is involutive (that is, $[D,D] \subset D$);*
2. *$D \cap \bar{D} = 0$ on $Z \backslash SM$ and $D \cap \bar{D} = \operatorname{span}_{\mathbb{C}} X$ on SM;*
3. *the fibres $\Sigma_x = Z \cap T_xM \cong \mathbb{D}$ are holomorphic (that is, $T^{0,1}\Sigma_x \subset D$).*

In particular, Z^{int} is a complex surface with $T^{0,1}Z^{\text{int}} = D$.

- Construction extends to other flows on SM (e.g. magnetic flows);
- Z is branched double cover of *classical twistor space* from Dubois-Violette (1983), O'Brian-Rawnsley (1985), LeBrun-Mason (2002).

Example: Suppose $M \subset \mathbb{C}$ with Euclidean metric, then

$$SM = \{(z, \mu) \in \mathbb{C}^2 : z \in M, |\mu| = 1\}.$$

Write $z = x + iy$ and $\mu = \cos\theta + i \sin\theta$, then

$$X = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y = \mu\partial_z + \bar{\mu}\partial_{\bar{z}} = \bar{\mu}\left(\mu^2\partial_z + \partial_{\bar{z}}\right).$$

Definition

On $Z = \{(z, \mu) \in \mathbb{C}^2 : z \in M, |\mu| \leq 1\}$ we define $D \subset T_{\mathbb{C}}Z$ by

$$D = \operatorname{span}_{\mathbb{C}}\left\{\mu^2\partial_z + \partial_{\bar{z}}, \partial_{\bar{\mu}}\right\}.$$

Say $f \in C^\infty(U)$ is *holomorphic* iff $(\mu^2\partial_z + \partial_{\bar{z}})f = \partial_{\bar{\mu}}f = 0$ on $U \subset Z$ open.

- $[D, D] = 0$ and $D \cap \bar{D} = \operatorname{span}_{\mathbb{C}} X$ for $|\mu| = 1$ are immediate;
- to incorporate different geometries/flows, replace X with $F = X + \lambda V$. If $\mu^2\lambda(z, \mu)$ is μ-holomorphic, then D is well defined by

$$D = \operatorname{span}_{\mathbb{C}}\left\{\mu^2\partial_z + \partial_{\bar{z}} + i\mu^2\lambda\partial_\mu, \partial_{\bar{\mu}}\right\};$$

- description in isothermal coordinates, but D is defined invariantly.

Notions of complex geometry (e.g. $\bar{\partial}$-complex, Dolbeaut cohomology, holomorphic vector bundles) are defined on Z *smooth up to the boundary.*

Let u_k be the kth vertical Fourier mode of a function u on SM.

$$\begin{aligned} \oplus_{k\geq k_0}\Omega_k &= \{u\in C^\infty(SM) : u_k = 0 \text{ for } k<k_0\};\\ \mho &= \{\mathbb{A}\in C^\infty(SM,\mathbb{C}^{n\times n}) : \mathbb{A}_k = 0 \text{ for } k<-1\}\end{aligned}$$

Theorem (Correspondence principles)

The twistor space of **any** *orientable Riemannian surface* (M,g) *satisfies:*

A) $H^{0,p}_{\bar\partial}(Z) \cong \begin{cases} \{u\in\oplus_{k\geq 0}\Omega_k : Xu=0\} & p=0,\\ \oplus_{k\geq -1}\Omega_k / X(\oplus_{k\geq 0}\Omega_k) & p=1,\\ 0 & p\geq 2.\end{cases}$

B) $\mathfrak{M}_n(Z) \equiv \left\{ \begin{array}{l} \textit{holomorphic vector bundle structures} \\ \textit{on } Z\times\mathbb{C}^n\textit{, up to isomorphism}\end{array}\right\} \cong \mho/\mathbb{G}.$

Theorem (TOG principle for simple surfaces)

- $H^{0,1}_{\bar\partial}(Z) \cong \mathfrak{M}_1(Z) = 0$ — Salo-Uhlmann *(2011)*
- $\mathfrak{M}_n(Z) = 0$ *for* $n\geq 2$ — B.-Paternain

Cohomology computations & TOG-principle suggest the following slogan:

The twistor space of a simple surface behaves like a (contractible) Stein surface.

Open questions:

- If $M, g)$ is simple, is Z^{int} an actual Stein surface?
- If (M, g_1) and (M, g_2) are both simple, do we have $Z_1 \cong Z_2$?
- Which holomorphic vector bundles exist in the non-simple case?

Thank you for your attention & happy birthday Gunther!

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Appendix: The blow down map β

Recall: The Cauchy Riemann equations on $Z(\mathbb{R}^2) \equiv \mathbb{C}_z \times \mathbb{D}_\mu$ are

$$(\mu^2 \partial_z + \partial_{\bar{z}}) f = 0 \quad \text{and} \quad \partial_{\bar{\mu}} f = 0.$$

The blow down map

The following map is holomorphic:

$$\beta \colon Z \to \mathbb{C}^2, \quad \beta(z, \mu) = (z - \mu^2 \bar{z}, \mu)$$

It has a partial inverse given by

$$\beta^{-1}(w, \mu) = \left(\frac{w}{1 + |\mu|^2} + \frac{2\,\mathrm{Re}(\bar{\mu} w)}{1 - |\mu|^4}, \mu \right), \quad (w, \mu) \in \beta(Z) \backslash \{|\mu| = 1\}.$$

- Original approach of Eskin-Ralston (2004) to obtain HIF: Use β to desingularise Z and apply the classical Oka-Grauert principle on $\beta(Z)$.